Reply to comment by R. L. Lysak on "Improved basis set for low frequency plasma waves"

P. M. Bellan[1]

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[1] In a Comment, R. L. Lysak argues against the validity of Bellan (2012) on the grounds that this paper uses fluid rather than kinetic theory. The Comment invokes a commonly used method for reducing the 3×3 wave equation matrix to a 2×2 matrix which then gives approximate dispersion relations. In this Response, it is shown that the same 3×3 wave equation matrix can be obtained from fluid theory and certain mathematical inconsistencies in the method of analysis used in the Comment are identified. It is shown that the dispersion relation derived in Bellan (2012) provides a much better description of the experimental observations reported by Kletzing et al. (2003) than does the dispersion relation proposed in the Comment and in Lysak and Lotko (1996).

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[2] The Comment on "Improved basis set for low frequency plasma waves" by P. M. Bellan will be referred to as L13. In L13, R. L. Lysak argues that the fluid theory used in *Bellan* [2012] (B12) is inadequate to describe the waves in question and that, instead, these waves must be described using kinetic theory. Then, using kinetic theory and referring to *Lysak and Lotko* [1996] (LL96), R. L. Lysak makes a great number of assumptions and approximations to arrive at a different result from B12. In response, we argue that wide-ranging regimes exist where kinetic theory and fluid theory correspond (much like wide-ranging regimes exist where quantum and classical mechanics correspond) and that the regimes under discussion are in such a range. We will demonstrate this correspondence by showing that the matrix equation advocated in L13 can be derived using fluid theory. The next issue is the mathematical approach used in L13. Certain approximations are made in L13 *before* taking the determinant of the matrix equation. This contrasts with B12 where the determinant of the matrix is calculated first without approximation. Because of intricate cancellations between various small quantities, it is demonstrated that errors result in L13 because approximations are made *before* evaluating the determinant rather than *after*. The remainder of this response will amplify on these two issues and will address the remarks in L13 in detail.

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1. Special Situation of $\beta = k_\parallel^2/k^2$

[3] While it is generally *incorrect* to assume that the fast mode is decoupled from the kinetic Alfvén mode as advocated by L13, it turns out that the fast mode is indeed decoupled in the special situation where $\beta = k_\parallel^2/k^2$. Examination of this special situation reveals how approximations in L13 cause L13 to miss important mode properties identified in B12. We thus temporally follow L13 by assuming the fast mode can be factored from equation (1) of L13 so a 2×2 matrix results but restrict consideration to the special situation where $\beta = k_\parallel^2/k^2$. Dropping displacement current but not dropping other terms, equation (1) of L13 becomes the 2×2 matrix

$$\begin{bmatrix} -\omega_{pi}^2/\left(\boxed{\omega^2} - \Omega_i^2\right) - n_\parallel^2 & n_\parallel n_\perp \\ n_\parallel n_\perp & \boxed{-\omega_{pi}^2/\omega^2} + 1/\left(k_\parallel^2 \lambda_{De}^2\right) - n_\perp^2 \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_z \end{bmatrix} = 0. \tag{1}$$

Boxes have been placed around the terms L13 drops to highlight certain algebraic issues. If these boxed terms are dropped as in L13, the determinant of equation (1) is

$$\left(\frac{\omega_{pi}^2}{\Omega_i^2} - n_\parallel^2\right)\left(\frac{1}{k_\parallel^2 \lambda_{De}^2} - n_\perp^2\right) - n_\parallel^2 n_\perp^2 = 0\,, \tag{2}$$

which yields the classic, cold-ion kinetic Alfvén wave dispersion

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = 1 + k_\perp^2 \rho_s^2, \tag{3}$$

where $\rho_s^2 = c_s^2/\omega_{ci}^2$ and $c_s^2 = \omega_{pi}^2 \lambda_{De}^2$. Let us now retain the terms dropped in L13. Upon multiplying through by

$\omega^2/k_\parallel^2 c^2$ and using $c^2/v_A^2 = \omega_{pi}^2/\Omega_i^2$ and $\beta = c_s^2/v_A^2$, equation (1) becomes

$$\begin{bmatrix} \frac{\omega^2}{k_\parallel^2 v_A^2}\frac{1}{\left(1-\omega^2/\Omega_i^2\right)} - 1 & \frac{k_\perp}{k_\parallel} \\ \frac{k_\perp}{k_\parallel} & \frac{\omega^2}{k_\parallel^2 v_A^2}\frac{\Omega_i^2}{\omega^2}\left(-1+\frac{\omega^2}{k_\parallel^2 v_A^2}\frac{1}{\beta}\right) - \frac{k_\perp^2}{k_\parallel^2}\end{bmatrix}\cdot\begin{bmatrix}E_x\\ E_z\end{bmatrix} = 0. \tag{4}$$

We now demonstrate that for the special case $k_\parallel^2/k^2 = \beta$, the exact root of equation (1) is $\omega^2 = k_\parallel^2 v_A^2$, and this is true for arbitrary ω/Ω_i. In other words, we demonstrate that the exact root is the cold plasma mode even though the temperature is finite. Using $\omega^2 = k_\parallel^2 v_A^2$ and $k_\parallel^2/k^2 = \beta$, we note that

$$-1 + \frac{\omega^2}{k_\parallel^2 v_A^2}\frac{1}{\beta} = -1 + \frac{k^2}{k_\parallel^2} = \frac{k_\perp^2}{k_\parallel^2}\,, \tag{5}$$

so with repeated use of $\omega^2/k_\parallel^2 v_A^2 = 1$, equation (4) can be expressed as

$$\begin{bmatrix} \frac{\omega^2}{\Omega_i^2}\frac{1}{1-\frac{\omega^2}{\Omega_i^2}} & \frac{k_\perp}{k_\parallel} \\ \frac{k_\perp}{k_\parallel} & \frac{\Omega_i^2}{\omega^2}\left(1-\frac{\omega^2}{\Omega_i^2}\right)\frac{k_\perp^2}{k_\parallel^2}\end{bmatrix}\cdot\begin{bmatrix}E_x\\ E_z\end{bmatrix} = 0. \tag{6}$$

On dividing the second row by $\frac{\Omega_i^2}{\omega^2}\left(1-\frac{\omega^2}{\Omega_i^2}\right)\frac{k_\perp}{k_\parallel}$, equation (6) becomes

$$\begin{bmatrix} \frac{\omega^2}{\Omega_i^2}\frac{1}{1-\frac{\omega^2}{\Omega_i^2}} & \frac{k_\perp}{k_\parallel} \\ \frac{\omega^2}{\Omega_i^2}\frac{1}{1-\frac{\omega^2}{\Omega_i^2}} & \frac{k_\perp}{k_\parallel}\end{bmatrix}\cdot\begin{bmatrix}E_x\\ E_z\end{bmatrix} = 0\,, \tag{7}$$

which has *zero* determinant since the second row is *identical* to the first row. Thus, we have established that $\omega^2 = k_z^2 v_A^2$ is an *exact* solution to equation (1) for arbitrary ω/Ω_i in the special situation where $k_\parallel^2/k^2 = \beta$. This important property is missed by L13 because the terms in boxes in equation (1) were dropped in L13. The condition $k_\perp v_A > \omega_{ci}$ corresponds to $k_\perp \rho_s > \beta^{1/2}$ and so is trivially satisfied in small β plasmas for equation (3) or its generalizations to be of any interest. Since ω/Ω_i is arbitrary, we have shown that $\omega^2 = k_\parallel^2 v_A^2$ is the exact solution to the system of equations in a situation where $\beta \ll 1$, $\omega/\Omega_i \ll 1$, and $k_\perp v_A > \omega_{ci}$, i.e., the regime assumed in L13. L13 denies the existence of this exact solution $\omega^2 = k_\parallel^2 v_A^2$ because L13 drops the terms in boxes in equation (1). L13 goes to the trouble of introducing modified Bessel functions and plasma dispersion functions when going from the 3×3 matrix of equation (1) of L13 to the 2×2 matrix of L13, but in the process of adding these more elaborate characterizations, L13 drops the $-\omega_{pi}^2/\omega^2$ term which was in the *zz* component of equation (1) of L13. It is seen that the lower right matrix element of equation (2) of L13 has no ion term, i.e., it does not have the $-\omega_{pi}^2/\omega^2$ term that was in the *zz* matrix element of equation (1) of L13; this was the lower right boxed term in equation (1) here.

2. Areas of Agreement With L13

[4] L13 is correct in stating that the caption of Figure 1 of LL96 was misread, and it is regretted that this misreading occurred. It is also agreed that the Bessel summation in equation (5) of L13 is correct.

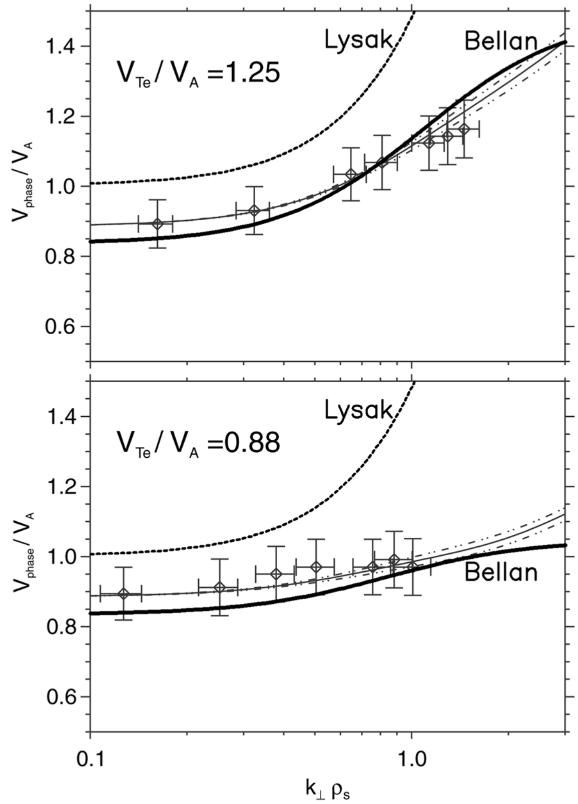

Figure 1. Reprint of Figure 3 from *Kletzing et al.* [2003] with, in addition, equation (12) from this Reply plotted as solid line labeled "Bellan" and also with equation (9) of L13 plotted as dotted line labeled "Lysak." It is assumed that $\omega/\omega_{ci} = 0.55$, $\gamma_i = 1.66$, $T_e = 3$ eV, and for best fit $T_i = 0.8$ eV. It should be noted that the ion temperature was not measured in *Kletzing et al.* [2003] and was presumed to be 1 eV based on measurements in similar experiments. (Reprinted from *Kletzing et al.* [2003, Figure 3] with permission. Copyright 2003 by the American Physical Society).

3. Reversion to Cold Plasma Character

[5] L13 states that "Another misleading statement in B12 is that the Alfvén mode reverts to its cold plasma character even if $m_e/m_i \ll \beta \ll 1$." This quotation is taken out of context and is misleading because B12 stated "When $\omega^2 = k^2 c_s^2$, the Alfvén mode decouples from the fast mode and reverts to its cold plasma character even if $m_e/m_i \ll \beta \ll 1$." L13 omitted the qualifying statement "When $\omega^2 = k^2 c_s^2$" and so incorrectly concluded that reversion to cold plasma character is a general result rather than a special situation occurring only when $\omega^2 = k^2 c_s^2$. This reversion to cold plasma character is seen in equation (10) of L13 since if $\omega^2 \to k^2 c_s^2$ the first term in square brackets on the left-hand side becomes infinite. In this situation the equation is solved by setting the second term in square brackets to zero, i.e., reverting to the cold plasma character with finite electron inertia included. The situation $\omega^2 = k^2 c_s^2$ and $\omega^2 = k_\parallel^2 v_A^2$ corresponds to $\beta = k_\parallel^2/k^2$, i.e., a situation where small quantity β equals small quantity $k_\parallel^2/k^2$. Because the regime in question has both

β small and $k_{\|}^2/k^2$ small, it is quite possible to have $\beta = k_{\|}^2/k^2$, and in fact, this situation occurs in the experiment reported by *Kletzing et al.* [2003].

4. Field-Aligned Current

[6] L13 falsely claims that field-aligned current does not appear in the basis set in B12. Field-aligned current actually appears more explicitly in the basis set $\{\mathbf{k}_\perp\cdot\tilde{\mathbf{J}}, \mathbf{k}_\perp\cdot\tilde{\mathbf{J}}\times\hat{z}\}$ used in B12 than in the approach advocated in L13. Specifically, the quasi-neutrality condition $\mathbf{k}\cdot\tilde{\mathbf{J}} = 0$ implies $\mathbf{k}_\perp\cdot\tilde{\mathbf{J}} = -k_z\tilde{J}_z$ so $\mathbf{k}_\perp\cdot\tilde{\mathbf{J}}$ is just the field-aligned current multiplied by the constant coefficient $-k_z$. Thus, equation (28) of B12 could trivially be written in terms of field-aligned current by replacing $\mathbf{k}_\perp\cdot\tilde{\mathbf{J}}$ by $-k_z\tilde{J}_z$, or, equivalently, the basis set could be expressed as $\{-k_z\tilde{J}_z, \mathbf{k}_\perp\cdot\tilde{\mathbf{J}}\times\hat{z}\}$.

5. Parallel Electric Field

[7] L13 claims that the parallel electric field is obscured. The parallel electric field is easily obtained in terms of $\{\mathbf{k}_\perp\cdot\tilde{\mathbf{J}}, \mathbf{k}_\perp\cdot\tilde{\mathbf{J}}\times\hat{z}\}$ from equation (24) in B12 since only the first and last terms on the right-hand side of equation (24) contribute as all other terms are perpendicular. The first term on the right-hand side of equation (24) is proportional to field-aligned current via $\mathbf{k}_\perp\cdot\tilde{\mathbf{J}}$, and the last term is proportional to $\mathbf{k}_\perp\cdot\tilde{\mathbf{J}}\times\hat{z}$ via equation (20).

6. Accounting for Gyromotion in Fluid Theory

[8] L13 asserts that Section 3.6 of *Krall and Trivelpiece* [1973] states that the pressure term in the generalized Ohm's law is not adequate to describe the effects of gyromotion. There is no statement of this sort in Section 3.6 of Krall and Trivelpiece. So long as the cyclotron orbit radius is small compared to the perpendicular wavelength, fluid theory describes the effects of gyromotion via diamagnetic drift (see Sec. 3.4 of *Chen* [1984]). As shown in Appendix A, equation (1) of L13 can be derived from two-fluid theory and this involves taking into account diamagnetic drift. Appendix A thus invalidates the claim in L13 that equation (1) is a kinetic theory result that differs from fluid theory.

7. Comparison of Dispersion Relations With Measurements in *Kletzing et al.* [2003]

[9] L13 claims that the LL96 dispersion relation has been verified by *Kletzing et al.* [2003]. However, the measurements reported by *Kletzing et al.* [2003] have $\omega/k_{\|}v_A$ less than unity for $k_\perp\rho_s \ll 1$, which contradicts the predictions of LL96. In order to fit the measurements, *Kletzing et al.* [2003] had to resort to a numerical solution of their equation (1) which is obtained from kinetic theory. They stated that, while being precise, their equation (1) is not intuitive and then present their equation (2) as being a fluid equation that while more intuitive is incapable of explaining $\omega^2/k_z^2v_A^2$ being less than unity. They state that "Lysak and Lotko have shown that (2) is often a very good approximation to the full kinetic approximation in the low frequency limit." We will now show that the fluid dispersion relation in B12 shows why $\omega^2/k_z^2v_A^2$ should be less than unity. We recall that B12 defined

$$\xi = \frac{\omega^2}{k^2v_A^2}, \beta = \frac{c_s^2}{v_A^2}, \Lambda = \frac{k^2v_A^2}{\omega_{ci^2}}, \alpha = \cos^2\theta \tag{8}$$

and note that Kletzing et al.'s data were a plot of $\omega/k_{\|}v_A$ versus $k_\perp\rho_s$ with ω/ω_{ci} held fixed. Equation (41) of B12 can be expressed in a form suitable for direct comparison with the experimental plots. Using

$$\xi\alpha^{-1} = \frac{\omega^2}{k_{\|}^2v_A^2}, \xi\Lambda = \frac{\omega^2}{\omega_{ci}^2}, \beta\Lambda = k^2\frac{c_s^2}{\omega_{ci}^2} \tag{9}$$

and

$$Q = 1 + \frac{k^2c^2}{\omega_{pe}^2} = 1 + k^2\rho_s^2\frac{v_A^2}{v_{Te}^2}, \tag{10}$$

equation (41) of B12 can be recast as

$$\frac{\omega^2}{k_{\|}^2v_A^2}\left(1 + k^2\rho_s^2\frac{v_A^2}{v_{Te}^2}\right) - 1 = k^2\frac{c_s^2}{\omega_{ci}^2} - \frac{\omega^2}{\omega_{ci}^2}, \tag{11}$$

which can be further rearranged to give

$$\frac{\omega}{k_{\|}v_A} = \sqrt{\frac{1 + k^2\rho_s^2\left(1 + \gamma_i\frac{T_i}{T_e}\right) - \frac{\omega^2}{\omega_{ci}^2}}{1 + k^2\rho_s^2\frac{v_A^2}{v_{Te}^2}}}. \tag{12}$$

Here $\rho_s^2 = \kappa T_e/m_i\omega_{ci}^2$, $v_{Te}^2 = \kappa T_e/m_e$, and $k^2 \simeq k_\perp^2$ are used. As shown in Figure 1, equation (12) provides a good fit to the experimental data presented in *Kletzing et al.* [2003]. Kletzing et al. are in the regime where $k_{\|}^2/k^2$ is of order β, and so the considerations presented in B12 are relevant. Because equation (12) is derived from the fluid theory given in B12, L13 argues that this equation is incorrect and instead one should use equation (9) of L13, with c_s^2/v_A^2 dropped on the grounds that β is negligible. With this assumption, equation (9) of L13 becomes

$$\frac{\omega}{k_{\|}v_A} = \sqrt{\frac{\mu_i}{1 - e^{-\mu_i}I_0(\mu_i)} + k_\perp^2\rho_s^2} \tag{13}$$

and so L13 is claiming that equation (13) is a more valid model than equation (12). We note that $\mu_i = k_\perp^2\rho_s^2T_i/T_e$ and $k^2 \simeq k_\perp^2$. The prediction of equation (13) is plotted as a dotted line in Figure 1. It is seen that equation (12) provides a much better fit to the data and also manifests the appropriate dependence on v_{Te}/v_A. Thus, the model presented in B12 does a much better job of describing actual waves than does the model in *Lysak and Lotko* [1996] (LL96). This is reasonable because B12 takes into account the dependence on ω/ω_{ci} whereas LL96 does not and because, as the maximum value of μ_i is 0.3 for the experimental data, the kinetic description of ions is not significantly different from the fluid description.

8. Limit of Small ω/ω_{ci}

[10] L13 and LL96 are based on the assumption that the *yz* and *zy* terms in the 3×3 matrix equation (e.g., equation (1) in L13) can be dropped when $\omega \ll \omega_{ci}$. By defining

$$X = \omega^2/k_{\|}^2v_A^2, Y = X/\beta, \varepsilon = \omega/\omega_{ci}, \alpha = k_{\|}^2/k^2 \tag{14}$$

and dropping displacement current (equivalent to assuming quasi-neutrality), this matrix equation can be written without further approximation as

$$\begin{bmatrix} \frac{X}{1-\varepsilon^2}-1 & i\frac{\varepsilon X}{1-\varepsilon^2} & \sqrt{\frac{1-\alpha}{\alpha}} \\ -i\frac{\varepsilon X}{1-\varepsilon^2} & \frac{X}{1-\varepsilon^2}-\frac{1}{\alpha} & -i\frac{X}{\varepsilon}\sqrt{\frac{1-\alpha}{\alpha}} \\ \sqrt{\frac{1-\alpha}{\alpha}} & i\frac{X}{\varepsilon}\sqrt{\frac{1-\alpha}{\alpha}} & (Y-1)\frac{X}{\varepsilon^2}-\frac{1-\alpha}{\alpha} \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0. \tag{15}$$

It is clearly seen that the yz and zy terms are of order ε^{-1} which diverges in the limit $\omega \ll \omega_{ci}$, so it is definitely not obvious that these yz and zy terms can be dropped. When the exact determinant of the matrix in equation (15) is evaluated, there are numerous intricate cancellations between the terms of order ε with the end result that the exact determinant is

$$(X-1)(XY\alpha - X - Y + 1) = \frac{\varepsilon^2}{\alpha}(Y\alpha - 1). \tag{16}$$

No assumptions regarding the size of ε have been made, and yet surprisingly, ε appears in only one place in equation (16). This exact determinant is equation (7) in B12. If one were to drop some terms of order ε in equation (15) while retaining others (e.g., if as in L13, one retains ε^2 in the zz matrix element, drops the yz and zy matrix elements altogether even though they scale as $1/\varepsilon$, and then drops ε everywhere else), one would obtain a result quite different from equation (16). Since one is seeking solutions where X is near unity, the result obtained using the methods advocated in L13 would be considerably different from equation (16). The intricate cancellations of ε leading to equation (15) suggest that there ought to be a better way of expressing the physical situation than equation (15); this better way is the improved basis set presented in B12.

9. On Making Approximations Before Taking Determinants

[11] L13 makes at least seven different approximations/assumptions (namely $\omega \ll \omega_{ci}$, $\omega \gg k_\parallel c_s$, $k_\perp v_A \gg \omega_{ci}$, $k_\parallel v_A \approx \omega$, $k_\perp \gg k_\parallel$, $\beta \ll 1$, and the assumption that the fast mode can be factored from the 3×3 matrix to obtain a 2×2 matrix). Some of these approximations/assumptions are made before taking the determinant and some made after. We now give a simple example showing how making approximations before taking a determinant can lead to error. Let ε and δ be two small parameters in the following "toy" problem. Suppose one wants to find solutions in the vicinity of $x = 1$ of the following equation

$$\begin{vmatrix} x-1 & \varepsilon & d \\ \varepsilon & x-b & 0 \\ d & 0 & \frac{1}{\delta^2}-g \end{vmatrix} = 0 \tag{17}$$

where b, d, g are of order unity. If one sets $\varepsilon = 0$ in analogy to the method in L13 and LL96, equation (17) reduces to

$$\begin{aligned} \begin{vmatrix} x-1 & 0 & d \\ 0 & x-b & 0 \\ d & 0 & \frac{1}{\delta^2}-g \end{vmatrix} &= (x-b)\begin{vmatrix} x-1 & d \\ d & \frac{1}{\delta^2}-g \end{vmatrix} \\ &= (x-b)\left((x-1)\left(\frac{1}{\delta^2}-g\right)-d^2\right) \\ &= 0. \end{aligned} \tag{18}$$

If one then approximates $\delta^2 g \ll 1$ because δ is small, then

$$(x-1)\frac{1}{\delta^2} - d^2 = 0 \tag{19}$$

so one obtains

$$x = 1 + d^2\delta^2. \tag{20}$$

However, if one starts again with equation (17) and takes the limit $\delta \to 0$ first, then the zz matrix element factors out, and the leading terms in the determinant are

$$\frac{1}{\delta^2}\left((x-1)(x-b)-\varepsilon^2\right) = 0 \tag{21}$$

which can be expressed as

$$x = 1 + \frac{\varepsilon^2}{x-b} \simeq 1 + \frac{\varepsilon^2}{1-b}, \tag{22}$$

which is completely different from equation (20). The exact determinant of equation (17) can be expressed as

$$x = 1 + \frac{d^2\delta^2}{(1-g\delta^2)} + \frac{\varepsilon^2}{(x-b)(1-g\delta^2)} - \frac{g\delta^2\varepsilon^2}{(x-b)\left(1-g\delta^2\right)}. \tag{23}$$

Using the assumptions that $x \simeq 1$ and that both ε and δ are small, equation (23) becomes

$$x = 1 + d^2\delta^2 + \frac{\varepsilon^2}{1-b}. \tag{24}$$

Clearly what counts is the ratio ε/δ. If one first sets $\varepsilon = 0$, when in fact ε and δ are the same order, an erroneous conclusion will result, namely, equation (20) instead of equation (22). If ε and δ are the same order, then both the δ^2 and the ε^2 term need to be retained. In the special case where $d = 1$, $b = 2$, and $\varepsilon = \delta$, the two small terms in equation (24) would cancel. This shows that the determinant should be evaluated before making approximations rather than the other way around.

10. Importance of *yz* and *zy* Matrix Elements

[12] It is argued both in LL96 and in the discussion of equation (6) of L13 that the yz and zy matrix elements can be discarded. We now provide a very simple demonstration that this is not so. In the limit of small $\omega \ll \omega_{ci}$ and dropping displacement, current equation (1) of L13 reduces to

$$\mathbf{M} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = \begin{bmatrix} \frac{\omega^2}{v_A^2}\frac{1}{1-\frac{\omega^2}{\omega_{ci}^2}} - k_\parallel^2 & 0 & k_\perp k_\parallel \\ 0 & \frac{\omega^2}{v_A^2} - k^2 & -\frac{\omega^2}{v_A^2} i \frac{\omega_{ci}}{\omega}\frac{k_\perp}{k_\parallel} \\ k_\perp k_\parallel & i\frac{\omega^2}{v_A^2}\frac{\omega_{ci}}{\omega}\frac{k_\perp}{k_\parallel} & \frac{\omega^2}{v_A^2}\left(\frac{\omega^2}{k_\parallel^2 c_s^2}-1\right)\frac{\omega_{ci}^2}{\omega^2} \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0, \tag{25}$$

where we have used

$$\lim_{\omega/\omega_{ci} \ll 1}\left[\frac{\omega^2}{v_A^2}\left(\frac{\omega^2}{k_\parallel^2 c_s^2}-1\right)\frac{\omega_{ci}^2}{\omega^2} - k_\perp^2\right] = \frac{\omega^2}{v_A^2}\left(\frac{\omega^2}{k_\parallel^2 c_s^2}-1\right)\frac{\omega_{ci}^2}{\omega^2} \tag{26}$$

to evaluate M_{zz}. The determinant of the matrix in equation (25) is

$$M_{xx}M_{yy}M_{zz} - M_{xz}M_{yy}M_{zx} - M_{xx}M_{yz}M_{zy} = 0. \tag{27}$$

$M_{xx}M_{yy}M_{zz}$ is of order $(\omega/\omega_{ci})^{-2}$ because the $(\omega/\omega_{ci})^{-2}$ scaling of M_{zz}. $M_{xx}M_{yz}M_{zy}$ is also of order ω_{ci}^2/ω^2 because of the $(\omega/\omega_{ci})^{-1}$ scaling of each of M_{yz} and M_{zy}. $M_{xz}M_{yy}M_{zx}$ is of order unity. Thus, equation (27) reduces to

$$M_{xx}(M_{yy}M_{zz} - M_{yz}M_{zy}) = 0, \tag{28}$$

which after some modest algebra is found to be exactly

$$(\omega^2 - k_z^2 v_A^2)\left[\omega^4 - \omega^2(k^2 v_A^2 + k^2 c_s^2) + k^2 k_z^2 v_A^2 c_s^2\right] = 0. \tag{29}$$

This is precisely Hirose's equation in the limit of $\omega \ll \omega_{ci}$, and furthermore it is the textbook ideal MHD result. This analysis shows that, contrary to Lysak's assertions, the M_{yz} and M_{zy} terms must not be dropped in the low frequency limit, i.e., in the limit where finite ω/ω_{ci} terms are discarded.

Appendix A: Derivation of Equation (11) in L13 Using Two-Fluid Theory

[13] The derivation of equation (1) in L13 used a full kinetic theory, and L13 considers this a kinetic model that is inherently superior to the fluid model in B12. We will now show that equation (1) in L13 can be derived from two-fluid theory and in particular identify the dynamical phenomena responsible for the matrix elements M_{yz} and M_{zy}. This demonstration that equation (1) in L13 can be derived from fluid theory shows that there is no difference between the kinetic and the fluid theory in the cold-ion, zero-electron-inertia regime associated with equation (1) in L13. We start by writing Ampere's law in the form

$$\nabla \times \tilde{\mathbf{B}} = \mu_0\varepsilon_0 \frac{\partial}{\partial t}\mathbf{K}\cdot\tilde{\mathbf{E}} \tag{A1}$$

where

$$\mathbf{K}\cdot\tilde{\mathbf{E}} = \tilde{\mathbf{E}} + \frac{1}{\varepsilon_0}\int^t \mathrm{d}t\tilde{\mathbf{J}} = \tilde{\mathbf{E}} - \frac{1}{i\omega\varepsilon_0}\tilde{\mathbf{J}} \tag{A2}$$

defines the dielectric tensor $\mathbf{K}$. Inserting equation (A1) into the curl of Faraday's law gives

$$\mathbf{k}(\mathbf{k}\cdot\tilde{\mathbf{E}}) - k^2\tilde{\mathbf{E}} + \frac{\omega^2}{c^2}\mathbf{K}\cdot\tilde{\mathbf{E}} = 0 \tag{A3}$$

or, in matrix form,

$$\begin{bmatrix} \frac{\omega^2}{c^2}K_{xx} - k_z^2 & \frac{\omega^2}{c^2}K_{xy} & \frac{\omega^2}{c^2}K_{xz} + k_x k_z \\ \frac{\omega^2}{c^2}K_{yx} & \frac{\omega^2}{c^2}K_{yy} - k^2 & \frac{\omega^2}{c^2}K_{yz} \\ \frac{\omega^2}{c^2}K_{zx} + k_z k_x & \frac{\omega^2}{c^2}K_{zy} & \frac{\omega^2}{c^2}K_{zz} - k_x^2 \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0. \tag{A4}$$

In order to determine $\mathbf{K}$, we consider the ion and electron contributions to $\tilde{\mathbf{J}}$ separately. In so doing, it is useful to note that the exact solution of the equation

$$\mathbf{v} = \mathbf{C} + \mathbf{v} \times F\hat{z} \tag{A5}$$

is

$$\mathbf{v} = C_z\hat{z} + \frac{\mathbf{C}_\perp}{1+F^2} - \frac{F\hat{z}\times\mathbf{C}_\perp}{1+F^2} \tag{A6}$$

as can be easily seen by dotting equation (A5) with $F\hat{z}$ and also crossing equation (A5) with $F\hat{z}$. Since the ions are assumed to be cold, the linearized ion equation of motion

$$-i\omega m_i\tilde{\mathbf{u}}_i = q_i(\tilde{\mathbf{E}} + \tilde{\mathbf{u}}_i \times \mathbf{B}) \tag{A7}$$

can be solved using equation (A6) to give

$$\tilde{\mathbf{u}}_i = \frac{iq_i}{\omega m_i}\left(\tilde{E}_z\hat{z} + \frac{\tilde{\mathbf{E}}_\perp}{1-\omega_{ci}^2/\omega^2} - \frac{i\omega_{ci}}{\omega}\frac{\hat{z}\times\tilde{\mathbf{E}}_\perp}{1-\omega_{ci}^2/\omega^2}\right). \tag{A8}$$

The ion current $\tilde{\mathbf{J}}_i = n_i q_i \tilde{\mathbf{u}}_i$ is thus

$$-\frac{1}{i\omega\varepsilon_0}\tilde{\mathbf{J}}_i = -\frac{\omega_{pi}^2}{\omega^2}\left(E_z\hat{z} + \frac{\tilde{\mathbf{E}}_\perp}{1-\omega_{ci}^2/\omega^2} - \frac{i\omega_{ci}}{\omega}\frac{\hat{z}\times\tilde{\mathbf{E}}_\perp}{1-\omega_{ci}^2/\omega^2}\right). \tag{A9}$$

Since $\omega^2 \ll k_z^2 v_{Te}^2$, the electrons are isothermal in which case the linearized electron equation of motion is

$$0 = n_e q_e(\tilde{\mathbf{E}} + \tilde{\mathbf{u}}_e \times \mathbf{B}) - i\mathbf{k}\tilde{n}_e \kappa T_e, \tag{A10}$$

where the last term is the linearized electron pressure and electron inertia has been dropped. The parallel component of equation (A10) gives

$$\tilde{n}_e = \frac{n_e q_e \tilde{E}_z}{\kappa T_e i k_z}. \tag{A11}$$

Two features of equation (A11) are worth noting: (1) In the electrostatic limit which is *not* being assumed here, equation (A11) would give the Boltzmann relation, and (2) even though equation (A11) comes from the parallel component of the electron equation of motion, equation (A11) does not prescribe $\tilde{u}_{ez}$. The perpendicular component of equation (A10) is

$$0 = n_e q_e(\tilde{\mathbf{E}}_\perp + \tilde{\mathbf{u}}_{\perp e} \times \mathbf{B}) - \tilde{n}_e \kappa T_e i k_x \hat{x}. \tag{A12}$$

Solving for $\tilde{\mathbf{u}}_{\perp e}$ gives

$$\tilde{\mathbf{u}}_{\perp e} = \frac{\tilde{\mathbf{E}}_\perp \times \mathbf{B}}{B^2} - \frac{\tilde{n}_e \kappa T_e i k_x \hat{x}\times\mathbf{B}}{n_e q_e}, \tag{A13}$$

where the second term is the electron diamagnetic drift and as discussed in Sec. 3.4 of *Chen* [1984] is a consequence of electron gyromotion. Using equation (A11) in equation (A12), it is seen that

$$\tilde{\mathbf{u}}_{\perp e} = \frac{\tilde{\mathbf{E}}_\perp \times \mathbf{B}}{B^2} + \frac{k_x}{k_z}\frac{\tilde{E}_z}{B}\hat{y}. \tag{A14}$$

The term involving k_x/k_z is independent of T_e but nevertheless results from T_e being finite. This is because the parallel component of equation (A10) involves a balance between the force due to the parallel electric field and the parallel pressure, while the perpendicular equation of motion contains a term involving the perpendicular pressure. Since the parallel and perpendicular pressures are the same except for a ratio k_x/k_z, the perpendicular pressure is just the parallel electric field multiplied by this ratio. The actual value of the temperature cancels (if temperature anisotropy existed, then k_x/k_z would be replaced by $k_x T_{e\perp}/k_z T_{e\parallel}$). The electron parallel velocity is determined using the linearized electron continuity equation

$$-i\omega\tilde{n}_e + ik_z n_e \tilde{u}_{ez} + ik_x n_e \tilde{u}_{ex} = 0. \tag{A15}$$

Using equation (A11) to give $\tilde{n}_e$ and equation (A14) to give

$$\tilde{u}_{ex} = \frac{\tilde{E}_y}{B}, \tag{A16}$$

equation (A15) can be solved for the parallel electron velocity

$$\tilde{u}_{ez} = -\frac{i\omega}{k_z^2}\frac{q_e}{\kappa T_e}\tilde{E}_z - \frac{k_x}{k_z}\frac{\tilde{E}_y}{B}. \tag{A17}$$

The electron current $\tilde{\mathbf{J}}_e = n_e q_e \tilde{\mathbf{u}}_{\perp e} + n_e q_e \tilde{u}_{ez}\hat{z}$ is therefore

$$-\frac{1}{i\omega\varepsilon_0}\tilde{\mathbf{J}}_e = -\frac{n_e q_e}{i\omega\varepsilon_0}\frac{\tilde{\mathbf{E}}_\perp \times \hat{z}}{B} - \frac{n_e q_e}{i\omega\varepsilon_0}\frac{k_x}{k_z}\frac{\tilde{E}_z}{B}\hat{y} + \left(\frac{\tilde{E}_z}{k_z\lambda_{De}^2} + \frac{n_e q_e}{i\omega\varepsilon_0}\frac{k_x}{k_z}\frac{\tilde{E}_y}{B}\right)\hat{z}. \tag{A18}$$

Using $n_e q_e/\varepsilon_0 B = -n_i q_i/\varepsilon_0 B = -\omega_{pi}^2/\omega_{ci}$, it is seen that

$$-\frac{1}{i\omega\varepsilon_0}\tilde{\mathbf{J}}_e = \frac{\omega_{pi}^2}{i\omega\omega_{ci}}\tilde{\mathbf{E}}_\perp \times \hat{z} + \frac{\omega_{pi}^2}{i\omega\omega_{ci}}\frac{k_x}{k_z}\tilde{E}_z\hat{y} + \left(\frac{\tilde{E}_z}{k_z^2\lambda_{De}^2} + i\frac{\omega_{pi}^2}{\omega\omega_{ci}}\frac{k_x}{k_z}\tilde{E}_y\right)\hat{z}. \tag{A19}$$

Combining equations (A9) and (A19) and using $\omega_{pi}^2/\omega_{ci}^2 = c^2/v_A^2$ gives

$$-\frac{1}{i\omega\varepsilon_0}\tilde{\mathbf{J}} = \frac{c^2}{v_A^2}\begin{bmatrix} \frac{1}{1-\frac{\omega^2}{\omega_{ci}^2}} & \frac{i\omega/\omega_{ci}}{1-\frac{\omega^2}{\omega_{ci}^2}} & 0 \\ -\frac{i\omega/\omega_{ci}}{1-\frac{\omega^2}{\omega_{ci}^2}} & \frac{1}{1-\frac{\omega^2}{\omega_{ci}^2}} & -i\frac{\omega_{ci}}{\omega}\frac{k_x}{k_z} \\ 0 & i\frac{\omega_{ci}}{\omega}\frac{k_x}{k_z} & \frac{\omega_{ci}^2}{k_z^2c_s^2} - \frac{\omega_{ci}^2}{\omega^2} \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix}. \tag{A20}$$

Inserting equation (A20) in equation (A2) and assuming $c^2/v_A^2 \gg 1$ (i.e., neglecting displacement current) gives

$$\mathbf{K} = \frac{c^2}{v_A^2}\begin{bmatrix} \frac{1}{1-\frac{\omega^2}{\omega_{ci}^2}} & \frac{i\omega/\omega_{ci}}{1-\frac{\omega^2}{\omega_{ci}^2}} & 0 \\ -\frac{i\omega/\omega_{ci}}{1-\frac{\omega^2}{\omega_{ci}^2}} & \frac{1}{1-\frac{\omega^2}{\omega_{ci}^2}} & -i\frac{\omega_{ci}}{\omega}\frac{k_x}{k_z} \\ 0 & i\frac{\omega_{ci}}{\omega}\frac{k_x}{k_z} & \left(\frac{\omega^2}{k_z^2c_s^2} - 1\right)\frac{\omega_{ci}^2}{\omega^2} \end{bmatrix}. \tag{A21}$$

Equation (A4) thus becomes

$$\begin{bmatrix} \frac{\omega^2}{v_A^2}\frac{1}{1-\frac{\omega^2}{\omega_{ci}^2}} - k_z^2 & \frac{\omega^2}{v_A^2}\frac{i\omega/\omega_{ci}}{1-\frac{\omega^2}{\omega_{ci}^2}} & k_xk_z \\ -\frac{\omega^2}{v_A^2}\frac{i\omega/\omega_{ci}}{1-\frac{\omega^2}{\omega_{ci}^2}} & \frac{\omega^2}{v_A^2}\frac{1}{1-\frac{\omega^2}{\omega_{ci}^2}} - k^2 & -\frac{\omega^2}{v_A^2}i\frac{\omega_{ci}}{\omega}\frac{k_x}{k_z} \\ k_zk_x & i\frac{\omega^2}{v_A^2}\frac{\omega_{ci}}{\omega}\frac{k_x}{k_z} & \frac{\omega^2}{v_A^2}\left(\frac{\omega^2}{k_z^2c_s^2} - 1\right)\frac{\omega_{ci}^2}{\omega^2} - k_x^2 \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0. \tag{A22}$$

Upon dividing by k_z^2, equation (A22) becomes equation (1) of L13. This procedure is clearly much more complicated than the derivation presented in B12 and yet, unlike B12, does not describe warm ions or finite electron inertia.

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